§1.1 - Introduction

"Geometry," meaning "measuring the earth," is the branch of math that has to do with spatial relationships. In other words, **geometry is a type of math used to measure things that are impossible to measure with devices**. For example, no one has been able take a tape measure around the earth, yet we are pretty confident that the circumference of the planet at the equator is 24,901.473 miles . How do we know that? The first known case of calculating the distance around the earth was done by Eratosthenes around 240 BCE. What tools do you think current scientists might use to measure the size of planets? The answer is geometry.

However, geometry is more than measuring the size of objects. If you were to ask someone who had taken geometry in high school what it is that s/he remembers, the answer would most likely be "proofs." (If you were to ask him/her what it is that s/he liked the least, the answer would probably be "proofs.") A study of Geometry does not have to include proofs. Proofs are not unique to Geometry. Proofs could have been done in Algebra or delayed until Calculus. The reason that High School Geometry almost always spends a lot of time with proofs is that the first great Geometry textbook, "The Elements," was written exclusively with proofs.

This textbook is based on Euclidean (or elementary) geometry. Euclidean geometry refers to a book written over 2,000 years ago called "The Elements" by a man named Euclid. In the book, Euclid started with some basic concepts. He built upon those concepts to create more and more concepts. His structure and method influence the way that geometry is taught today. Euclid's book and interpretations of it were used as part of the curriculum of many high schools even until the beginning of the 20th century. Although this textbook is not a re-interpretation of *The Elements*, it will include more than just facts about geometric objects; the ability to "prove" that a particular answer is correct is part of the course.

§1.2 - Reasoning

There are two general ways of reaching conclusions: inductive reasoning and deductive reasoning.

- **Inductive Reasoning**

Inductive reasoning is what we use most often. Inductive reasoning is reaching a conclusion based on previous observations. For example, if I notice that the sun rises in the east every day, then through inductive reasoning I could conclude that the sun will rise from the East tomorrow. In math, we may notice a pattern from which we draw conclusions. Look at the following pattern:

$1^2 = 1$	$1 \leq 1$
$2^2 = 4$	$2 \leq 4$

$$\begin{array}{cc} 3^2 = 9 & 3 \leq 9 \\ (-1)^2 = 1 & -1 \leq 1 \\ (-2)^2 = 4 & -2 \leq 4 \end{array}$$

Through inductive reasoning, we might conclude that whenever a number is squared, the result is a number which is greater than or equal to the original number. Based on observations, this appears to be true. Inductive logic is not certain, though. Looking at the example above, you may have already surmised that there are some numbers for which our conclusion does not hold:

$$\left(\frac{1}{2}\right)^2 = \frac{1}{4} \qquad \frac{1}{2} > \frac{1}{4}$$

The same can be applied to problems outside of Math. A beginning observer of American baseball may conclude, after watching several games, that the game is over after 9 innings. He will only realize that this observation is false after observing a game which is tied after 9 innings. Inductive reasoning is useful but not certain. There will always be a chance that there is an observation that will show the reasoning to be false. Only one observation is needed to prove the conclusion to be false.

Much of the reasoning in geometry is like this, consisting of three simple stages (see example A):

1. Look for commonalities
   A pattern.
2. Make a *conjecture*-
   An unproven statement that you will prove.
3. Prove/Disprove
   The conjecture.

- ### Deductive Reasoning

Deductive reasoning is reaching a conclusion by combining known truths to create a new truth. Unlike inductive reasoning, deductive reasoning is certain, provided that the normal rules of logic are used to conclude such truths. In order to use deductive reasoning there must be a starting point, those are normally called the axioms or postulates of the theory. For example, an axiom in geometry asserts that given two points there is only one line that contains both points. Observe that while this is an axiom, it can be used to deduce that two different lines intersect in at most one point.

Not only axioms can be used to deduce new truths. Other knowledge deduced from the axioms using the rules of logic can be used to validate new truths. For example, we can conclude that if three points A, B and C are not in the same line then the lines determined by two of them can only meet at A, B and C (since we already know that two lines can only intersect at one point, all that is necessary to prove is that the lines determined by

two of the three points are different, and that is immediate since the given points do not belong to any one line).

- **Vocabulary**

conjecture: An unproven statement that is based on observations.

Examples of Reasoning

Example A: Making a Conjecture

Complete this conjecture:
The sum of the first *x* odd positive integers is _?_

Solution - Inductive:
sum of the first 1 odd positive integers: 1 = 1 = 1^2
sum of the first 2 odd positive integers: 1 + 3 = 4 = 2^2
sum of the first 3 odd positive integers: 1 + 3 + 5 = 9 = 3^2
sum of the first 4 odd positive integers: 1 + 3 + 5 + 7 = 16 = 4^2
sum of the first 5 odd positive integers: 1 + 3 + 5 + 7 + 9 = 25 = 5^2
sum of the first 6 odd positive integers: 1 + 3 + 5 + 7 + 9 + 11 = 36 = 6^2

And so on...

The sum of the first *x* odd positive integers is *x*^2.

Solution - Deductive

Prove that the sum of the first n odd numbers: $\sum_{x=1}^{n}(2x-1) = n^2$.

$$\begin{aligned}
&\sum_{x=1}^{n}(2x-1) \\
&= 2\sum_{x=1}^{n} x - \sum_{x=1}^{n} 1 \\
&= 2\frac{n(n+1)}{2} - n \\
&= n^2 + n - n \\
&= n^2
\end{aligned}$$

Exercises

1) *All vegetables are good for you. Broccoli is a vegetable. Therefore, broccoli is good for you.* This is an example of what type of reasoning?

2) *Broccoli is a vegetable. Broccoli is green. Therefore, all vegetables are green.* Why is this conclusion invalid?

§ 1.3 - Undefined Terms

In Geometry, there are three undefined terms: **points**, **lines**, and **planes**. Although most terms in geometry are defined based on previously defined terms, it is impossible to

define every geometric term this way. The first geometric term cannot be defined based on previously defined terms.

Although we cannot formally define these three terms, we can informally describe them. We also use these terms to help us write definitions of other terms such as segment, or ray. There is no axiom that says that **lines are drawn straight**. What this means, is that the definition of line depends on the theory that you are studying, so in Hyperbolic Geometry a line does not look like a line in Euclidean Geometry, since they are defined differently.

In Euclidean Geometry, a **point is thought of as having neither breadth, nor width, nor height.** Now imagine taking a very sharp pencil, and making a dot on a piece of paper. Now imagine looking at it under a magnifying glass, the dot would be big, and we would be able to see it has a height and a breadth. A point is not a dot, because a point would have neither height nor breadth, but we can imagine that in the very middle of the dot is a point.

With the non-definitions out of the way, let's look at how these things work. A point is usually represented by a dot on a piece of paper. A point is useful because it tells us *exactly* where something is, and we can then build observations, conjectures, and rules from that information. For example, we can say that two points determine a line. What this means is that once you know where two points are, you know where the line that contains both of the points must be. Notice that if you only know where one point is, there are an infinite number of lines that can contain that one point, and if you know where three points are, there is a pretty good chance that there isn't any single line that would contain all three points.

In Euclidean Geometry, a **line is thought of as having length but neither width, nor height.** A line is such that any two points on the line describe the *shortest distance* between those two points. Lines also carry on forever in both directions. Imagine a piece of string, hold the two ends and pull them tight. The string represents the shortest distance between the two ends. Remember though that a line does not have any width or height. Under a magnifying glass we see the string has width. A tightly drawn string is not a line, because a line would not have a width, but we can imagine a line in the exact middle of the string.

Now usually when we talk about a line in geometry we mean a straight line as described above, but there are other lines in Euclidean geometry, called curves. Curves are not straight. The circumference of a circle is an example of a curve. (We will get to circles later in the syllabus).

Okay, we have talked about lines, now, so how do they behave? We usually represent a line by drawing it on a piece of paper using a ruler to connect the points and extending it past the points. We can take pieces of a line and call them **line segments** (more on that later in the book) and we can cross two lines and get both a point (where they intersect)

and some **angles** (more on those later too). We can also choose to ignore half of a line by cutting it off at a point and calling what we have left a **ray** (again, more later).

A plane has two dimensions: width and length. Both of these dimensions are infinite, and, because there are only two dimensions, a **plane is perfectly flat and infinitely thin, meaning it has no thickness dimension.** Because of this, a plane doesn't really have a top or a bottom because whatever is on the top is also on the bottom. If you take two planes and make them intersect, you get a line (more on that later) and if you take three points that are not all in the same line, there is only one plane that can contain all three (more on that later too). Planes are useful because a plane can hold all of the two dimensional (flat) shapes that geometry uses. We usually think of one side of a piece of paper (or a computer screen) as part of a plane. While this is not exactly correct, like the representations of a point and a line, this is useful.

§1.4 - Axioms/Postulates

A postulate or **axiom is a statement which is taken to be self-evident, and cannot be proved.** They are the starting point from which any system in Mathematics (such as geometry) is built up. The axioms of geometry state properties of points, lines, and planes that are consistent with our intuitive understanding of them. For example, one axiom states that given two points there is a unique line that passes through those two points (a property of incidence between points and lines). In Euclidean geometry, there are five axioms:

1. A straight line segment can be drawn joining any two points.
2. Any straight line segment can be extended indefinitely in a straight line.
3. Given any straight line segment, a circle can be drawn having the segment as its radius and one endpoint as its center.
4. All right angles are congruent.
5. Given a line and a point off the line, exactly one new line can be drawn through the point that is parallel to the given line.

From these postulates we can deduce all the theorems of Euclidean geometry.

§1.5 – Theorems

A theorem is a proposition that is deducible from basic axioms.

Postulates:

1) Between any two points, there exists one and only one line.

2) If two lines intersect, then their intersection is a point.

3) Given any 3 non-collinear points, there is exactly one plane that can be constructed which will include all of them.

4) If two planes intersect, then their intersection is a line.

Exercises

1) Draw a point on a piece of paper. How many lines can you draw through that point?

2) Draw two points on a piece of paper. How many lines can you draw through both points?

3) Draw three points on a piece of paper. How many lines can you draw through all three points? Why? What undefinable object could connect all three points? Is there a way to draw the points so that a line goes through all three?

Vocabulary

- **Inductive Reasoning** - process of reasoning in which the assumption of an argument supports the conclusion, but does not ensure it
- **Deductive Reasoning** - process of reasoning in which the argument supports the conclusion based upon a rule
- **Conjecture** - a mathematical statement which has been proposed as a true statement, but which no one has yet been able to prove or disprove
- **Theorem** - a proposition that has been or is to be proved on the basis of explicit assumptions
- **Hypothesis** - a proposed explanation which can be a proposition ("A causes B")
- **Postulate** - a mathematics statement which is used but cannot be proven
- **Axiom** - a formal logical expression used in a deduction to yield further results

§4.1 Translations, Reflections and Rotations

Before we continue, you need to know what Translations, Reflections, and Rotations are. Let us start with the following image of a triangle.

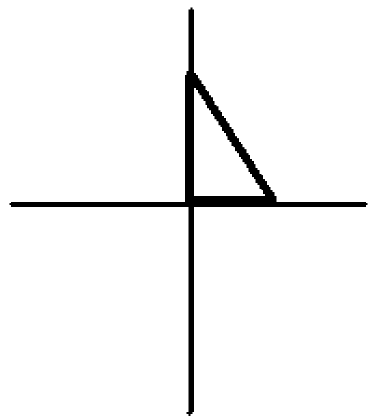


Image I.

Translation means moving the image horizontally (along the x axis: Image II) or vertically (along the y axis: Image III).

These are examples of translations from Image I.

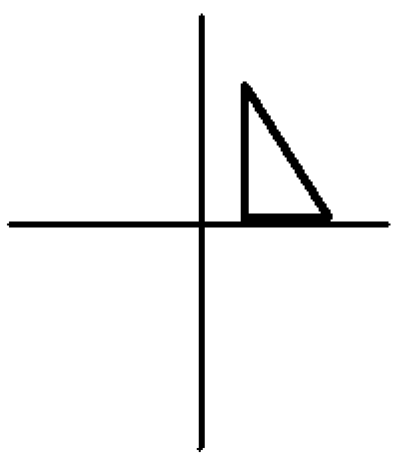


Image II

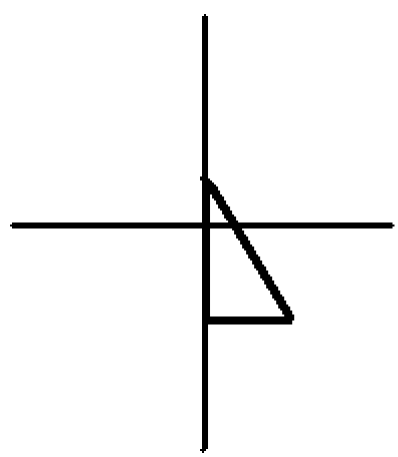
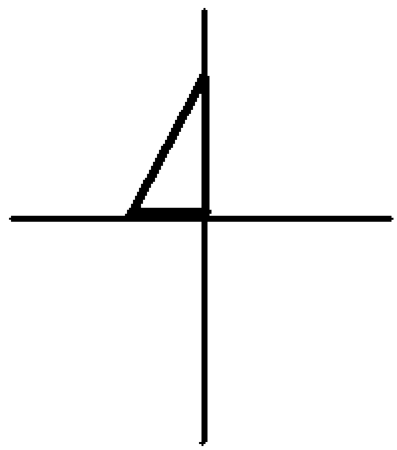


Image III

Reflection means flipping the image either over the x axis (a horizontal line: Image V) or over the y axis (a vertical line: Image IV).

These are examples of reflection from Image I.



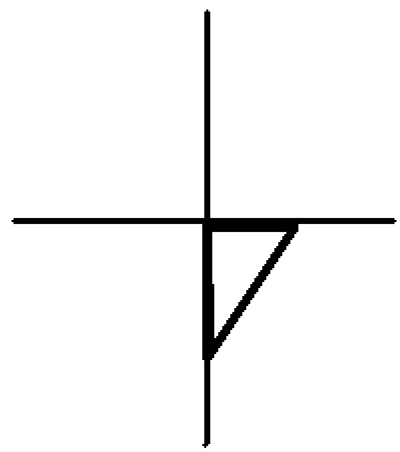


Image IV Image V

Rotation means moving the image from a pivot point.

Here, Image VI is a example of Image I rotated 90°.

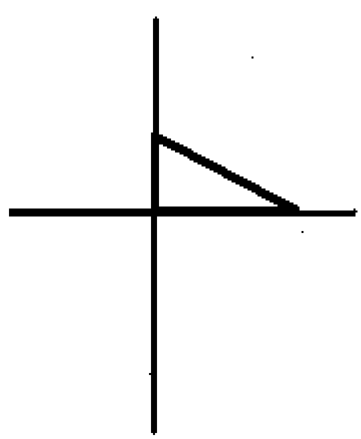


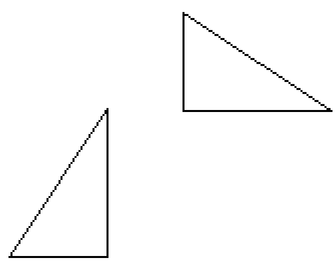
Image VI

Notice that in all of the operations performed on the triangle above, none of the operations changed the angles of the triangle, or the lengths of any of the line segments. In all of the operations shown above, the only things that change are the location of the three points that make up the triangle.

Translations, reflections and rotations do not fundamentally change a shape. Terms for shapes that undergo any of these transformations are covered in the next section.

§4.2 Congruence and Similarity

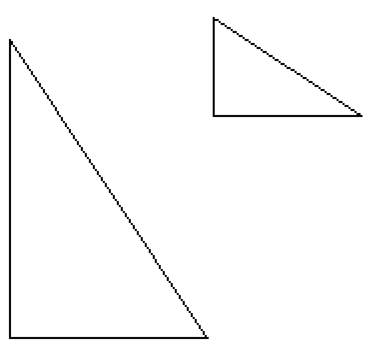
Intuitively, congruent **shapes are shapes that are exactly the same**. Technically speaking, two shapes are congruent if you can translate, rotate and/or reflect one of them in such a way that it coincides exactly with the other shape. Hence, a shape may be translated, reflected or rotated and remain congruent to its counterpart.



Congruent triangles

These two triangles above are congruent, even though they are rotations of each other.

Similar shapes are shapes that, when scaled, are exactly the same. A shape may be translated, reflected or rotated and remain similar to its counterpart. In a sense, **similar shapes are scale models of each other, that is, they are proportional.**



Similar triangles

These triangles are similar because when one is scaled down and rotated, it becomes congruent with the other.

Vocabulary

- **Congruent Shapes** - Shapes that coincide exactly when translated, reflected, and/or rotated.
- **Similar Shapes** - Shapes that coincide exactly when translated, reflected, rotated and/or scaled.

Exercises

1) On graph paper, draw a triangle with the points (0, 0), (0, 15) and (15, 0). Draw the triangle reflected over the Y axis (the vertical axis). Draw the triangle translated up 25 units.

2) On graph paper, draw a square with the points (0, 0), (0, 15), (15, 15) and (15, 0). When you scale this shape by any amount, what happens to the point at (0, 0)?

3)On graph paper, draw a square with the points (0, 0), (0, 6), (6, 6) and (6, 0). When you scale this shape by 2, what happens to the point at (0, 6)?
